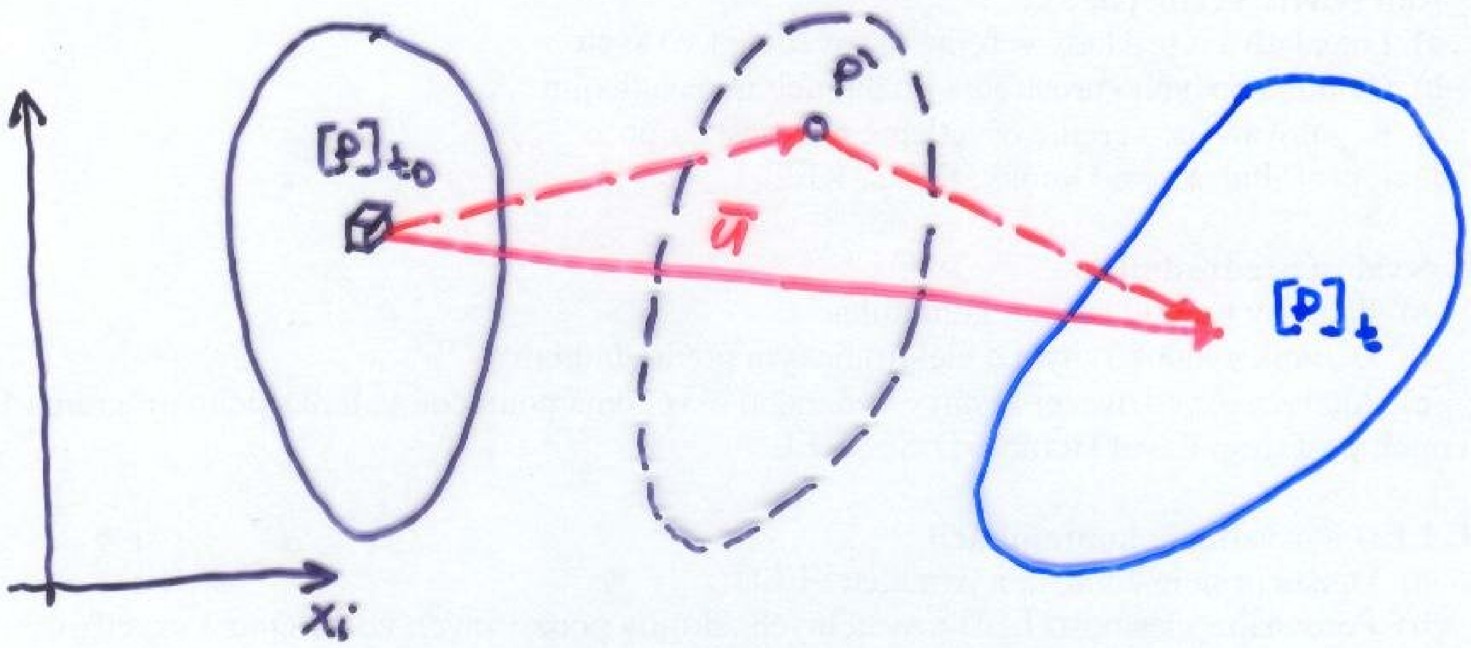
TENZOR DEFORMÁCIE

KINEMATIKA PODDAJNÉHO KONTINUA

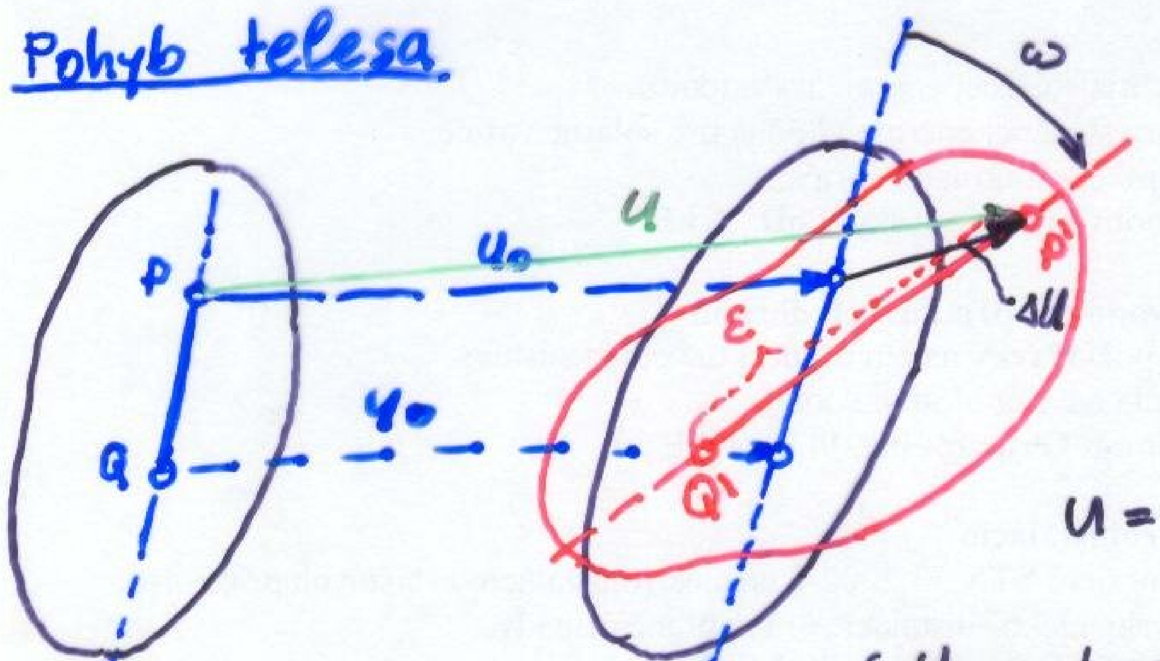
① Pohyb telesa, vektor posunutia



$\bar{u}$ – vektor posunutia hmotného bodu

u_i – vektorové pole posunutia bodov telesa

Pohyb telesa



$u = u_0 + \Delta u$

$\Delta u = f(\omega, \varepsilon)$

$u = u_0 \oplus \omega \oplus \varepsilon$

tuhý pohyb $\left\{ \begin{array}{l} u_0 \text{ – translácia} \\ \omega \text{ – tuhá rotácia} \end{array} \right.$

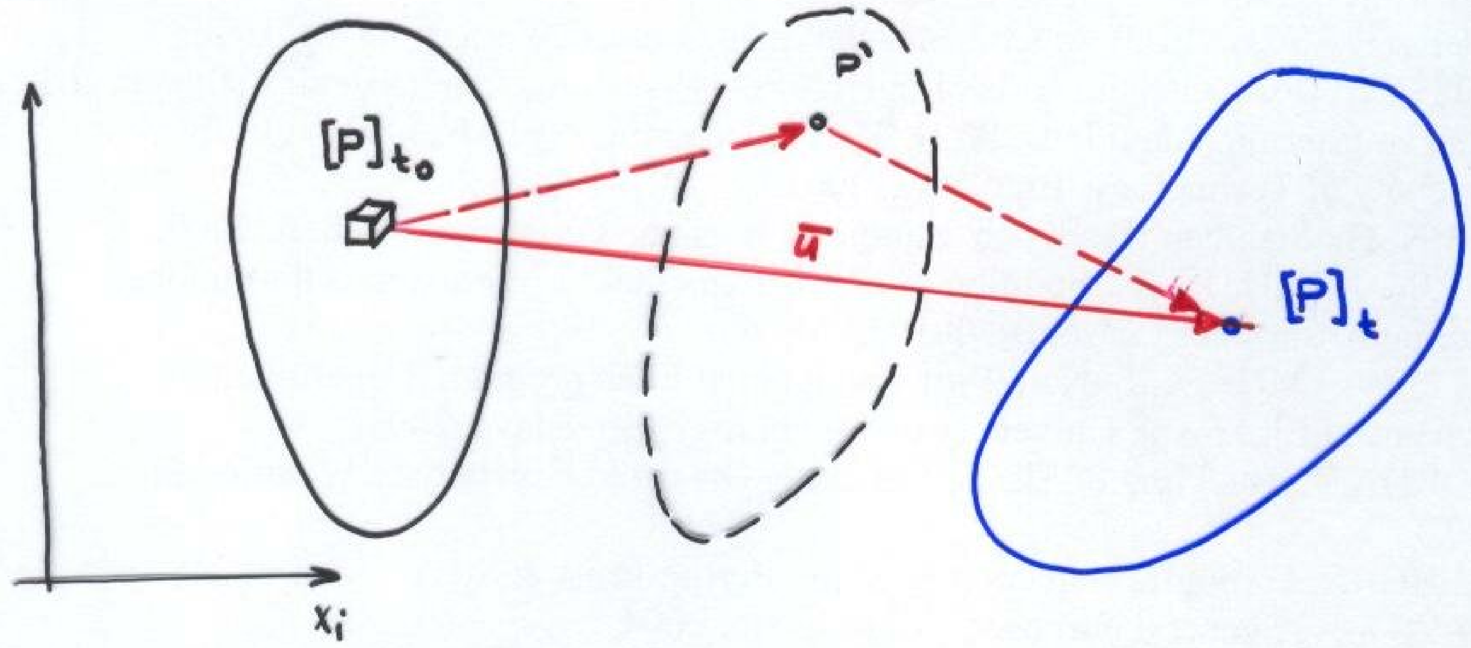
ε – pretvorenie

Úloha: Určiť u a jeho zložky !!

TENZOR DEFORMÁCIE

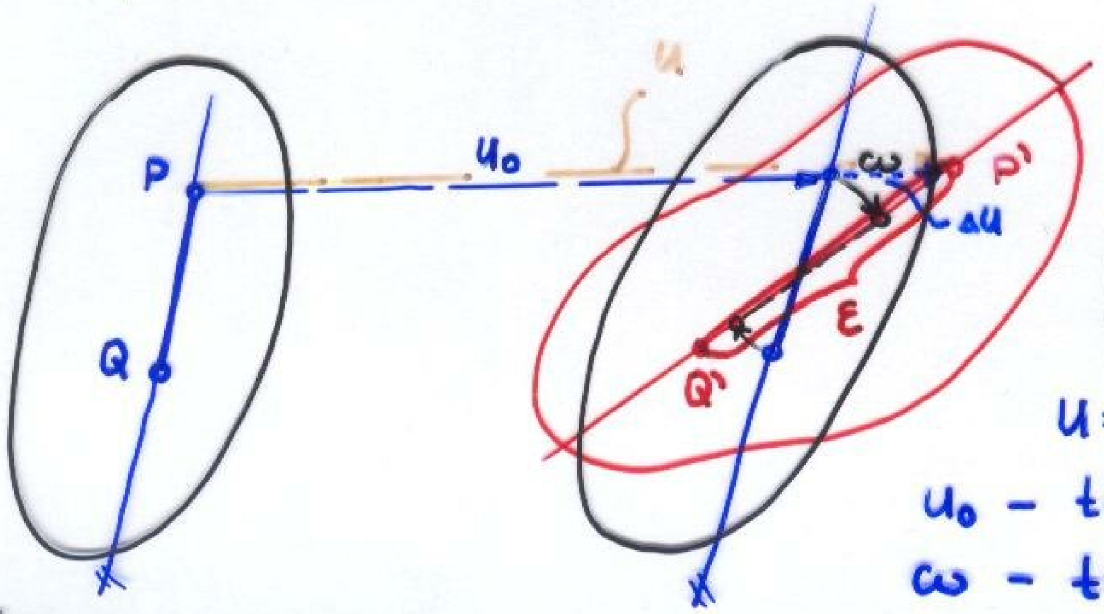
KINEMATIKA PODDAJNÉHO KONTINUA

① Mikro – a makroskop. prístup, pohyb telesa, vektor posunutia



$\bar{u}$ – vektor posunutia (vektorové pole posunutí bodov)

Pohyb telesa



formálny zápis

$u = u_0 + \Delta u$

$\Delta u = f(\omega, \varepsilon)$

$$u = u_0 + \omega + \varepsilon$$

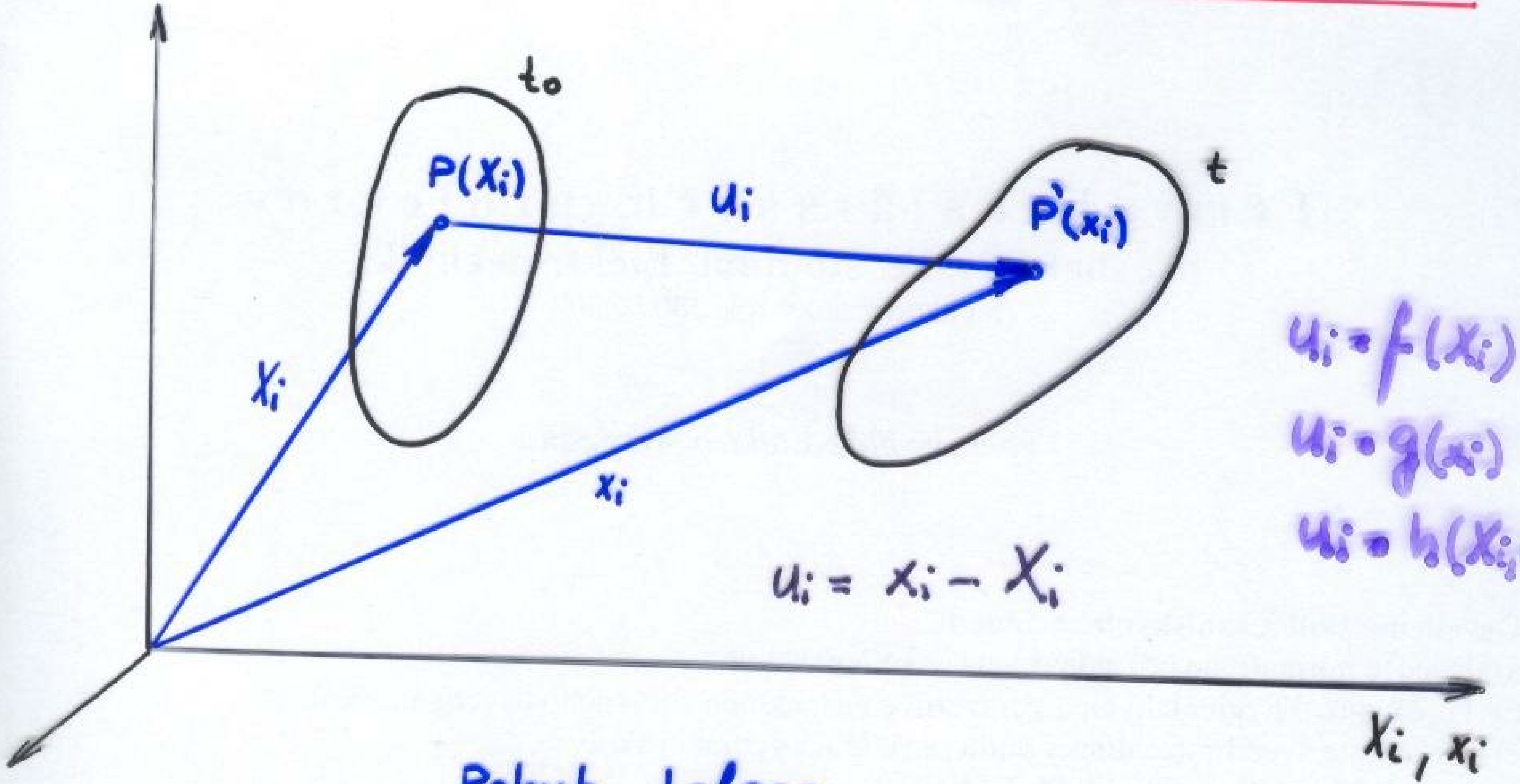
u_0 – translacia
ω – tuhá rotácia
} tuhý pohyb

ε – pretvorenie

Úloha:

Určiť u a jeho zložky

② Lagrangeovské a Eulerovské súradnice, gradient deformácie a posunutia



$$u_i = x_i - X_i$$

$$\left.\begin{aligned} u_i &= f(X_i) \\ u_i &= g(x_i) \end{aligned}\right\} ?$$
$$u_i = h(X_i, x_i)$$

Pohyb telesa

$X_i \equiv (X_1, X_2, X_3)$ – pôvodné – Lagrangeovské súrad.

$x_i \equiv (x_1, x_2, x_3)$ – okamžité – Eulerovské súradnice

<u>Vektor posunutia</u> : $u_i = x_i - X_i$

a, Lagrangeovský spôsob opisu pohybu

$x_i = x_i(X_1, X_2, X_3, t)$ X_i – nezávislé premenné

$u_i = u_i(X_j, t) = x_i(X_j, t) - X_i$

Gradient posunutia:

$$u_{i,j} = \frac{\partial u_i}{\partial X_j} = \mathrm{grad}\,\bar{u} = \frac{\partial x_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij}$$

$$\frac{\partial x_i}{\partial X_j} = \mathrm{grad}\,\bar{x} = F_L$$ – gradient deformácie (Lagr.)

Poznámka: Čas ako reálnu veličinu neuvažujeme

b, Eulerovský spôsob opisu pohybu telesa

$$X_i = X_i(x_1, x_2, x_3, t)$$ x_i – nezávislé premenná

$$u_i = u_i(x_j, t) = x_i - X_i(x_j, t)$$

Gradient posunutia:

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}$$

Gradient deformácie:

$$\frac{\partial X_i}{\partial x_j} = \mathrm{grad}\,\bar{X} = F_E$$

Použitie: LS ⇒ elastostatika, elastodynamika, medzné stavy, konečné deformácie ⇒ počiatočná konfigurácia je známa

ES $\Rightarrow$ hydrodynamika, prúdenie, časovo závislé – tuhoplastické pretvorenia

$$F \equiv \frac{\partial X_i}{\partial x_j} = ?$$

- Zmiešaná formulácia spája výhody LS a ES
- Pre $\varepsilon \ll 1$ ($\bar{u} \ll 1$, $\Delta\bar{u} \ll 1$, $\omega \ll 1$) je $x_i \equiv X_i \Rightarrow$ LS $\equiv$ ES

3, Pretvorenie telesa

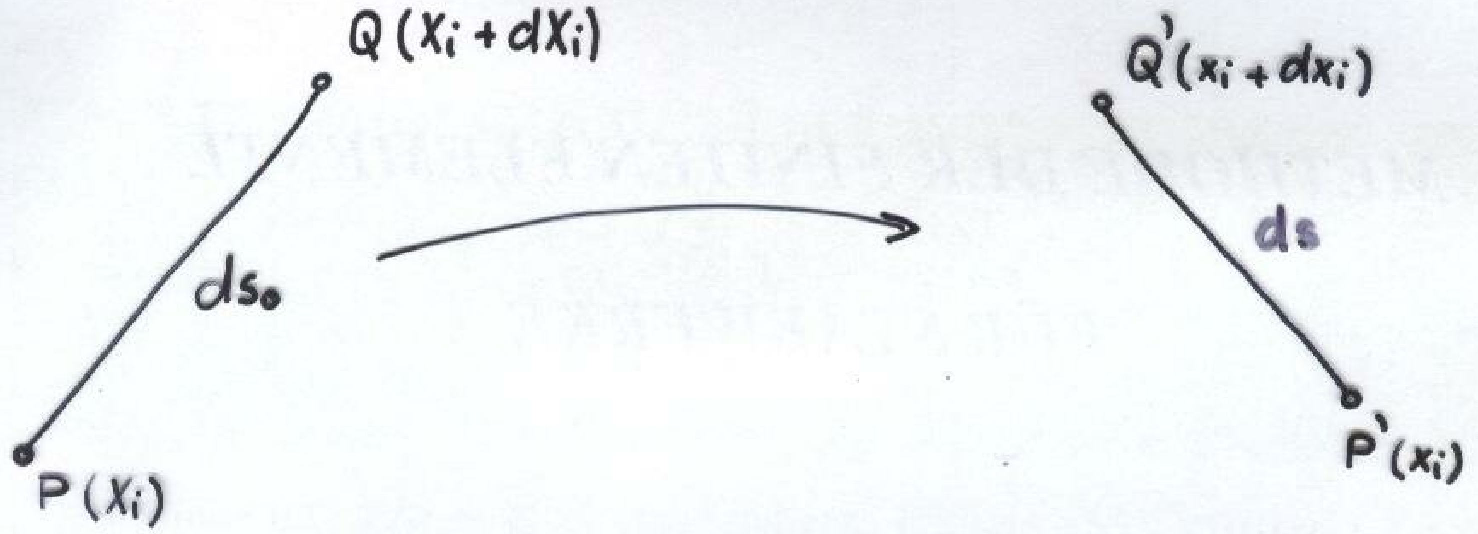
Vektor posunutia $= f$(tuhý pohyb + pretvorenie)

Napätie $\neq f$(tuhého pohybu), $\sigma = f(\varepsilon)$

$\Rightarrow$ Uplatnenie záverov trhacej skúšky pre analýzu pretvorenia telesa

Teleso sa deformuje, ak sa zmení vzdialenosť vybraných hmotných bodov. V opačnom prípade teleso koná tuhý pohyb.

Analýza pretvorenia $\equiv$ skúmanie zmeny dĺžkového úseku $ds_0 \rightarrow ds$



$$\left[\overline{PQ}\right]^2 = ds_0^2 = dX_1^2 + dX_2^2 + dX_3^2 = dX_i\, dX_i = \delta_{ij}\, dX_i\, dX_j$$

$$\left[\overline{P'Q'}\right]^2 = ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i\, dx_i = \delta_{ij}\, d\underbrace{\cancel{X}_i}_{x_i}\, d\underbrace{\cancel{X}_j}_{x_j} = \delta_{ij}\, dx_i\, dx_j$$

Pričom: $\quad dx_i = \dfrac{\partial x_i}{\partial X_j}\, dX_j \quad$ – Lagrange

$$dX_i = \frac{\partial X_i}{\partial x_j}\, dx_j \quad \text{– Euler}$$

Potom:

$$ds_0^2 = \delta_{ij} \frac{\partial X_i}{\partial x_l}\, dx_l\, \frac{\partial X_j}{\partial x_m}\, dx_m = \delta_{ij}\, dx_l\, dx_m \frac{\partial X_i}{\partial x_l} \frac{\partial X_j}{\partial x_m}$$

$$ds^2 = \delta_{ij} \frac{\partial x_i}{\partial X_l} \cdot \frac{\partial x_j}{\partial X_m}\, dX_l\, dX_m$$

Zaujíma nás:

$$ds^2 - ds_0^2 = ?$$

a, Lagrangeov spôsob

$$ds^2 - ds_0^2 = \underbrace{\delta_{ij}\, dx_i\, dx_j}\; - \delta_{ij}\, dX_i\, dX_j$$

$$x_i = x_i(X_j)$$

$$ds^2 - ds_0^2 = \delta_{ij} \frac{\partial x_i}{\partial X_l} \frac{\partial x_j}{\partial X_m} dX_l\, dX_m - \delta_{ij}\, dX_i\, dX_j$$

Premenujeme $\quad i = l$

$\qquad\qquad\quad\; j = m$

$\leftrightarrow$ (navzájom)

$$ds^2 - ds_0^2 = \delta_{lm} \frac{\partial x_l}{\partial X_i} \frac{\partial x_m}{\partial X_j} dX_i\, dX_j - \delta_{ij}\, dX_i\, dX_j =$$

$$= \underbrace{\left[\delta_{lm} \frac{\partial x_l}{\partial X_i} \frac{\partial x_m}{\partial X_j} - \delta_{ij}\right]}_{2E_{ij}} dX_i\, dX_j = 2E_{ij}\, dX_i\, dX_j$$

$$ds^2 - ds_0^2 = 2E_{ij}\, dX_i\, dX_j$$

$$E_{ij} = \frac{1}{2}\left[\delta_{lm} \frac{\partial x_l}{\partial X_i} \frac{\partial x_m}{\partial X_j} - \delta_{ij}\right]$$

Green-Lagrange

tenzor konečných deformácií

b) Eulerov spôsob

$$X_i = X_i(x_j)$$

$$ds^2 - ds_0^2 = \delta_{ij}\, dx_i\, dx_j - \delta_{ij} \frac{\partial X_i}{\partial x_l} \frac{\partial X_j}{\partial x_m} dx_l\, dx_m =$$

$l \leftrightarrow i$, $m \leftrightarrow j$ zámena indexov

$$= \underbrace{\left(\delta_{ij} - \delta_{lm} \frac{\partial X_l}{\partial x_i} \frac{\partial X_m}{\partial x_j}\right)}_{2e_{ij}} dx_i\, dx_j = 2 e_{ij}\, dx_i\, dx_j$$

$$e_{ij} = \frac{1}{2}\left[\delta_{ij} - \delta_{lm} \frac{\partial X_l}{\partial x_i} \frac{\partial X_m}{\partial x_j}\right]$$

Cauchy - Euler
Almansi
tenzor KD

c) Vlastnosti E_{ij} a e_{ij}

1) $E_{ij} = E_{ji}$, $e_{ij} = e_{ji}$ – symetričnosť

2) invariantnosť voči tuhému pohybu (objektívnosť)

$\Rightarrow$ Ak $ds^2 - ds_0^2 = 0 \Rightarrow E_{ij} = 0$

$e_{ij} = 0$

3) $e_{ij} = E_{ij}$ pre $\varepsilon \ll 1$
$x_i \doteq X_i$

④ Zložky E_{ij} a e_{ij} v karte'zskom SS

a, Green- Lagrangeov tenzor

$$E_{ij} = \frac{1}{2}\left[\delta_{lm}\frac{\partial x_l}{\partial X_i}\frac{\partial x_m}{\partial X_j} - \delta_{ij}\right] = \frac{1}{2}\left[\frac{\partial x_m}{\partial X_i}\frac{\partial x_m}{\partial X_j} - \delta_{ij}\right]$$

$$\delta_{lm}\frac{\partial x_l}{\partial X_i} = \frac{\partial}{\partial X_i}\left(\delta_{lm} x_l\right) = \frac{\partial}{\partial X_i} x_m = \frac{\partial x_m}{\partial X_i}$$

$$u_i = x_i - X_i \Rightarrow x_i(X_j) = u_i + X_i$$ $(x_m = u_m + X_m)$

$$\frac{\partial x_m}{\partial X_i} = \frac{\partial u_m}{\partial X_i} + \delta_{mi} = u_{m,i} + \delta_{mi}$$

$$\frac{\partial x_m}{\partial X_j} = u_{m,j} + \delta_{mj}$$

$\delta_{mi}\, u_{m,j} = u_{i,j}$

$\delta_{mj}\, u_{m,i} = u_{j,i}$

$\delta_{mj}\,\delta_{mi} = \delta_{ji} = \delta_{ij}$

napr.:

$$E_{ij} = \frac{1}{2}\left[(u_{m,i} + \delta_{mi})(u_{m,j} + \delta_{mj}) - \delta_{ij}\right] =$$

$$= \frac{1}{2}\left[u_{m,i}\, u_{m,j} + u_{j,i} + u_{i,j} + \delta_{ij} - \delta_{ij}\right]$$

$$E_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{m,i}\, u_{m,j}\right]$$

napr.:

$$E_{11} = \frac{1}{2}\left[u_{1,1} + u_{1,1} + u_{m,1}\, u_{m,1}\right] = u_{1,1} +$$

$$+ \frac{1}{2}\left[u_{1,1} . u_{1,1} + u_{2,1}\, u_{2,1} + u_{3,1}\, u_{3,1}\right] =$$

$$= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right]$$

$$E_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) + \frac{1}{2}\left(u_{1,1}\, u_{1,2} + u_{2,1}\, u_{2,2} + u_{3,1}\, u_{3,2}\right)$$

atď. ...

b, Almansiho tenzor deformácie

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \delta_{lm} \frac{\partial X_l}{\partial x_i} \frac{\partial X_m}{\partial x_j}\right) = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_m}{\partial x_i} \frac{\partial X_m}{\partial x_j}\right)$$

$$u_m = x_m - X_m \Rightarrow X_m = x_m - u_m$$

$$\Rightarrow \frac{\partial X_m}{\partial x_i} = \delta_{mi} - u_{m,i} \quad , \quad \frac{\partial X_m}{\partial x_j} = \delta_{mj} - u_{m,j}$$

Potom:

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - (\delta_{mi} - u_{m,i})(\delta_{mj} - u_{m,j})\right) =$$

$$= \left(\delta_{ij} - (\delta_{ij} - u_{i,j} - u_{j,i} + u_{m,i}\, u_{m,j})\right)$$

$$e_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} - u_{m,i}\, u_{m,j}\right]$$

Napr.: $$e_{11} = u_{1,1} - \frac{1}{2}\left((u_{1,1})^2 + (u_{2,1})^2 + (u_{3,1})^2\right) =$$

$$= \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right]$$

Tenzor nekonečne malých deformácií

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$x_i \equiv X_i \quad (u_{i,j})^2 = 0$

$\Rightarrow e_{ij} \equiv E_{ij} \equiv \varepsilon_{ij}$

$$(\varepsilon_{ij}) = \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\ & \text{sym} & \\ & & u_{3,3} \end{pmatrix}$$

Cauchy (Eulerov) tenzor nekonečne malých deformácií

$\varepsilon_{ij} = \varepsilon_{ji}$ - sym. tenzor

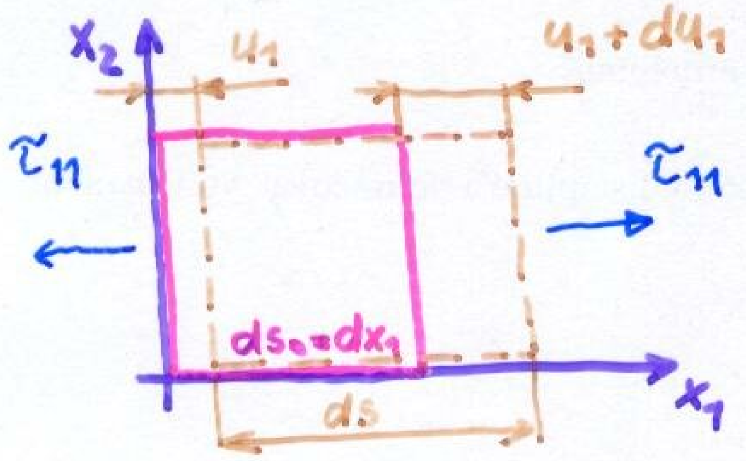
GEOMETRICKÝ VÝZNAM TENZORA DEFORMÁCIE

a, Tenzor nekonečne malých deformácií

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Napr.: $i,j = 1,2$ - rovinná deformácia

$$\varepsilon_{11} = \frac{1}{2}(u_{1,1} + u_{1,1}) = u_{1,1} = \frac{\partial u_1}{\partial x_1}$$



$$ds - ds_0 = du_1 = \frac{\partial u_1}{\partial x_1} dx_1$$

$$\frac{ds - ds_0}{dx_1} = \frac{\partial u_1}{\partial x_1} = \varepsilon_{11} = \frac{du_1}{dx_1}$$

Ak $ds_0 \equiv dx_1$

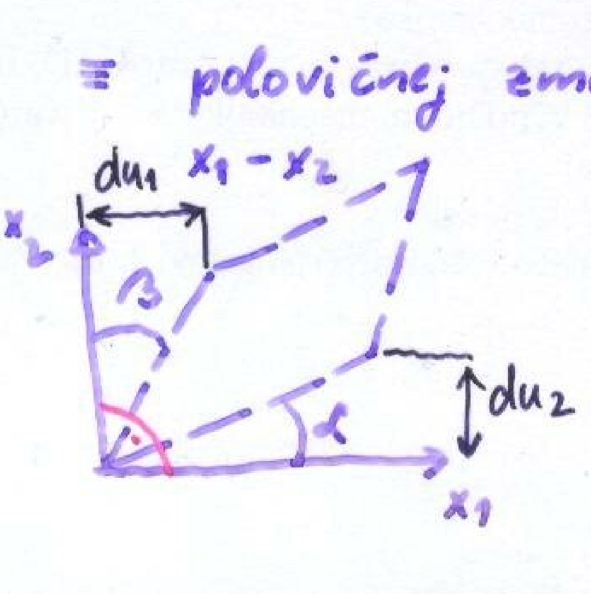
$$\Rightarrow \varepsilon_{11} = \frac{ds - ds_0}{ds_0}$$ - pomerné predĺženie v smere x_1

$\Rightarrow \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ - pomerné predĺženia

Podobným spôsobom možno určiť geom. význam zmiešaných zložiek (Teória pružnosti).

Napr.: $$\varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$\equiv$ polovičnej zmene pravého uhla EH v rovine $x_1 - x_2$



$$\varepsilon_{12} = \frac{1}{2}(\alpha + \beta)$$

$$\frac{du_1}{dx_2} = \operatorname{tg}\beta \doteq \beta \qquad \frac{du_2}{dx_1} = \operatorname{tg}\alpha \doteq \alpha$$

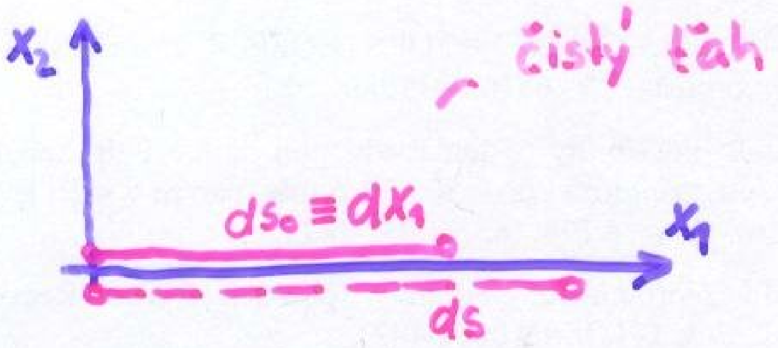
b) Tenzor konečných deformácií E_{ij}

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}\right)$$

Napr.:

$$ds^2 - ds_0^2 = 2E_{ij}\,dX_i\,dX_j$$

Nech $i = j = 1 \Rightarrow 2E_{11}\,dX_1\,dX_1 = ds^2 - ds_0^2$



$$\Rightarrow E_{11} = \frac{1}{2}\,\frac{ds^2 - ds_0^2}{(dX_1)^2} = \frac{1}{2}\,\frac{ds^2 - ds_0^2}{ds_0^2} = \frac{(ds - ds_0)\overbrace{(ds + ds_0)}^{2ds_0}}{2\,ds_0^2}$$

$\Rightarrow$ Za predpokladu nekonečne malých deformácií:

$$E_{11} = \frac{(ds - ds_0)\cdot\cancel{2}\cancel{ds_0}}{\cancel{2}\,ds_0^2} = \frac{ds - ds_0}{ds_0} = \varepsilon_{11}$$

Pre malé deformácie ($\varepsilon < 0{,}001$) ma E_{ij} význam ε_{ij}

Pri konečných deformáciach nemá fyzikálny význam

$\Rightarrow \quad \varepsilon = \varepsilon^{e} + \varepsilon^{pl} \qquad E = E^{e} + E^{pl} = ?$

NÁZOV: T. D.

ČÍSLO: 1

PREDMET: MK

ROČNÍK:

ČÍSLO ZLOŽKY

Tenzor nekonečne malých rotácií

Rozdeľme gradient posunutia ($u_{i,j}$ - dyáda) na symetrickú a antisymetrickú časť:

$$u_{i,j} = \underbrace{\frac{1}{2}(u_{i,j} + u_{j,i})}_{\varepsilon_{ij}} + \underbrace{\frac{1}{2}(u_{i,j} - u_{j,i})}_{\omega_{ij}}$$

ε_{ij} - tenzor malých deformácií

ω_{ij} - antisymetrický tenzor (tenzor rotácie)

$$\omega_{ij} = \begin{pmatrix} \emptyset & \frac{1}{2}(u_{1,2} - u_{2,1}) & \frac{1}{2}(u_{1,3} - u_{3,1}) \\ -\omega_{12} & \emptyset & \frac{1}{2}(u_{2,3} - u_{3,2}) \\ -\omega_{13} & -\omega_{23} & \emptyset \end{pmatrix}$$

NÁZOV: TD

PREDMET: MK

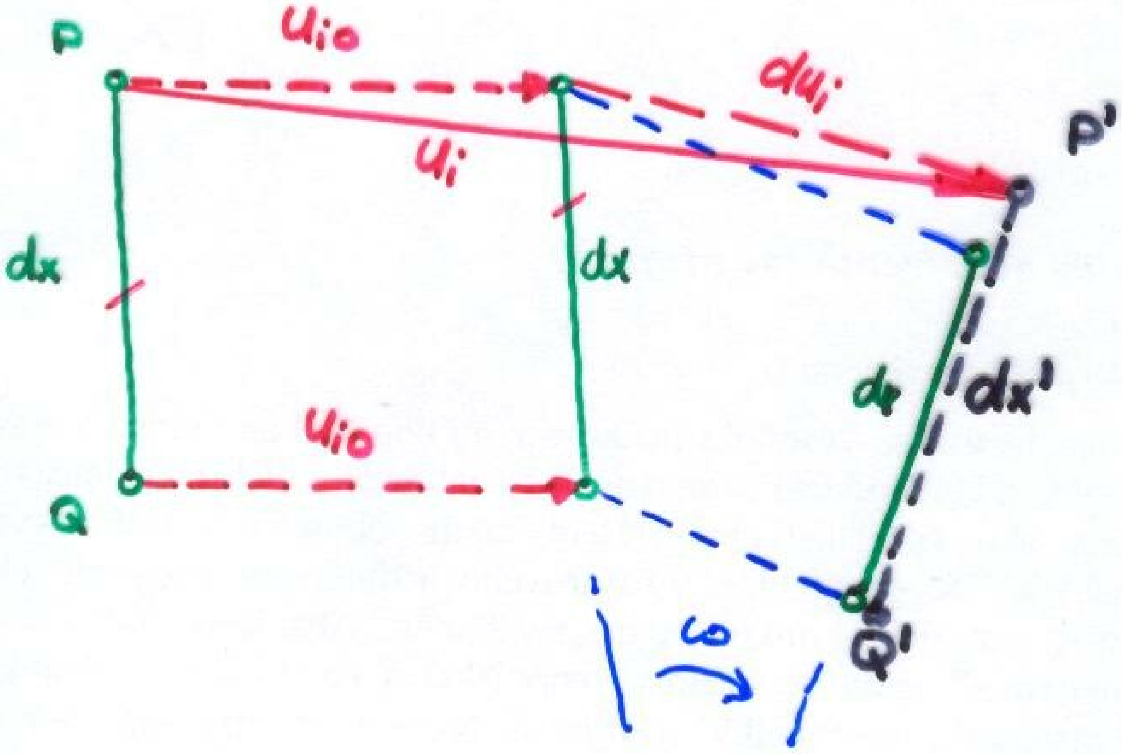
ROČNÍK:

ČÍSLO: 2

ČÍSLO ZLOŽKY

Zložky pohybu pri deformácii telesa:

Deformačný proces = tuhý pohyb + pretvorenie



Vektor posunutia bodu $P \rightarrow P'$

$$u_i = u_{io} + du_i$$

u_{io} - translačný tuhý pohyb

Prirastok posunutia:

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j = u_{i,j}\, dx_j = \underbrace{\varepsilon_{ij}\, dx_j}_{du_i^{\varepsilon}} + \underbrace{\omega_{ij}\, dx_j}_{du_i^{\omega}}$$

ε_{ij} predstavuje predĺženie (pretvorenie) $dx_i \rightarrow dx_i'$

Čo predstavuje ω_{ij} = ?

NÁZOV: TD

PREDMET: MK

ROČNÍK:

ČÍSLO: 3

ČÍSLO ZLOŽKY

Nech $dx \to dx'$ tuhým pohybom $\Rightarrow$

$$dx = dx' \Rightarrow \varepsilon_{ij} = \emptyset$$

$\Rightarrow$ Vektor posunutia

$$du_i = \omega_{ij}\, dx_j$$

$\Rightarrow$ ω_{ij} predstavuje tuhý pohyb (rotáciu elementu dx.

HLAVNÉ POMERNÉ PREDĹŽENIA

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right) = \varepsilon_{ji}$$

$\Rightarrow$ symetrický tenzor

$\Rightarrow$ Rovnica vlastného tvaru:

$$\left(\varepsilon_{ij} - \lambda\,\delta_{ij}\right) v_j = 0 \Rightarrow \lambda^{(k)} \begin{cases} \lambda^{(1)} = \varepsilon_1 \\ \lambda^{(2)} = \varepsilon_2 \\ \lambda^{(3)} = \varepsilon_3 \end{cases} \text{hlavné pomerné predĺženia}$$

Invarianty deformácie:

$$I_1 = tr\, \varepsilon_{ij} = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$I_2 = \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}\right)$$

$$I_3 = e_{ijk}\,\varepsilon_{i1}\,\varepsilon_{j2}\,\varepsilon_{k3}$$

$$HSS^{\tau_{ij}} = HSS^{\varepsilon_{ij}}$$

NÁZOV: TD

PREDMET: MK

ROČNÍK:

ČÍSLO: 4

ČÍSLO ZLOŽKY

INÉ MIERKY PRETVORENIA

- gradient posunutia : $u_{i,j}$

- Cauchyho tenzory :

$$\bar{C}_{ij} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} \quad \text{resp.} \quad C_{ij} = \frac{\partial X_k}{\partial x_i} \frac{\partial X_k}{\partial x_j}$$

- Fingerove tenzory :

$$B_{ij} = \frac{\partial X_i}{\partial x_k} \frac{\partial X_j}{\partial x_k} \qquad \bar{B}_{ij} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k}$$

Tieto tenzory nie sú invariantné voči tuhému pohybu :

Nech $x_i = X_i$

$$C_{ij} = \delta_{ki} \delta_{kj} = \delta_{ij} \neq \phi \quad \Rightarrow \quad B_{ij} = \delta_{ij}$$

NÁZOV: TD

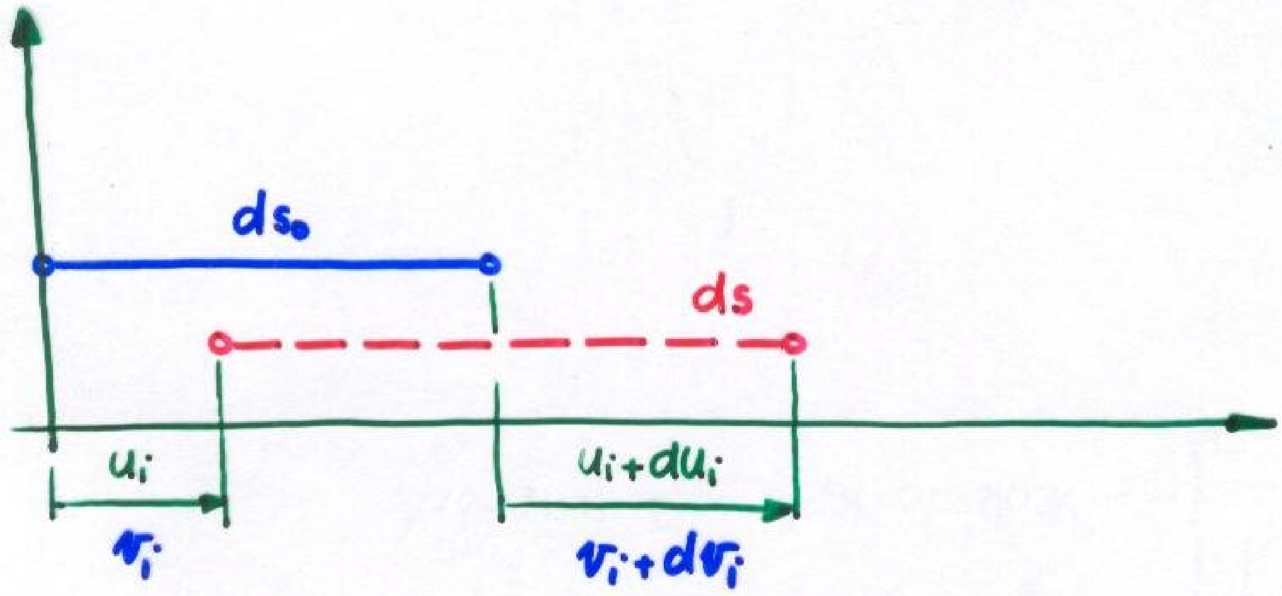
PREDMET: MK

ROČNÍK:

ČÍSLO: 5

ČÍSLO ZLOŽKY

RÝCHLOSŤ POSUNUTIA A DEFORMÁCIE



$x_i \neq x_i(t) = const$

$u_i = u_i(x_j, t) \quad \Rightarrow \quad v_i = \dfrac{\partial u_i}{\partial t} = \dot{u}_i = v_i(t)$

$dv_i = \dfrac{\partial v_i}{\partial x_j} dx_j = \underbrace{\tfrac{1}{2}(v_{i,j} + v_{j,i})}_{V_{ij}} dx_j + \underbrace{\tfrac{1}{2}(v_{i,j} - v_{j,i})}_{\Omega_{ij}} dx_j$

$(du_i = \varepsilon_{ij} dx_j + \omega_{ij} dx_j)$

$d\dot{u}_i = \dot{\varepsilon}_{ij} dx_j + \dot{\omega}_{ij} dx_j = dv_i$

$dv_i = V_{ij} dx_j + \Omega_{ij} dx_j$

$V_{ij} = \dot{\varepsilon}_{ij}$ – tenzor rýchlosti deformácie

$\varepsilon_{ij} = \dot{\varepsilon}_{ij}\, dt$

$\omega_{ij} = \dot{\omega}_{ij}\, dt$

$\Omega_{ij} = \dot{\omega}_{ij}$ – tenzor rýchlosti rotácie